

A probabilistic approach to consecutive pattern avoiding in permutations *

Guillem Perarnau

Departament de Matemàtica Aplicada IV.
Universitat Politècnica de Catalunya, BarcelonaTech.
 guillem.perarnau@ma4.upc.edu

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Abstract

We present a new approach to the problem of enumerating permutations of length n that avoid a fixed consecutive pattern of length m . We use this idea to give explicit upper and lower bounds on the number of permutations avoiding a pattern of length m . As a corollary, we obtain a simple proof of the CMP conjecture [7], regarding the most avoided pattern, recently shown by Elizalde [6]. Finally, we also show that most of the patterns behave similar to the least avoided one.

Keywords: Permutations, Consecutive pattern avoiding, Monotone patterns, CMP conjecture, Probabilistic method.

1 Introduction

Let \mathcal{S}_n be the symmetric group of permutations of length n . Consider a permutation $\pi = (\pi_1, \dots, \pi_n) \in \mathcal{S}_n$ and a *pattern* $\sigma = (\sigma_1, \dots, \sigma_m) \in \mathcal{S}_m$. Henceforth m will be a fixed integer while n will be considered to tend to infinity. For any sequence of different positive integers $X = (x_1, \dots, x_k)$, we define the *standardization* of X , $\text{st}(x_1, \dots, x_k)$, as the permutation in \mathcal{S}_k obtained by relabeling each element of X on the set $\{1, \dots, k\}$ such that the order among all the elements of X is preserved.

A permutation $\pi \in \mathcal{S}_n$ *contains* σ *as a consecutive pattern* if there exists $0 \leq i \leq n - m$ such that $\text{st}(\pi_{i+1}, \dots, \pi_{i+m}) = \sigma$, this is, there are m consecutive elements in π that have the relative order prescribed by σ . For instance, if $\sigma = (12 \dots m)$, π contains σ as a consecutive pattern if and only if it contains m consecutive increasing elements (a run of length m). A pattern $\pi \in \mathcal{S}_n$ is called *σ -avoiding* if it does not contain σ as a consecutive pattern. Denote by $\alpha_n(\sigma)$, the number of permutations in \mathcal{S}_n that are σ -avoiding.

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The problem of determining $\alpha_n(\sigma)$ is inspired by the problem of finding the number of permutations of length n that avoid a pattern σ non necessarily in consecutive positions. A permutation $\pi \in \mathcal{S}_n$ *contains* σ if there exist $1 \leq i_1 < \dots < i_m \leq n$ such that $\text{st}(\pi_{i_1}, \dots, \pi_{i_m}) = \sigma$. Clearly, if π avoids σ , then π also avoids σ as a consecutive pattern. Knuth [11] introduced the latter problem and exactly determined the number of permutations avoiding some pattern of length 3. There are many interesting results in the area (see e.g. [3, 1]) as well as the famous Stanley-Wilf conjecture which was solved by Marcus and Tardos [12].

To provide an exact formula for $\alpha_n(\sigma)$ is a tough problem when n becomes large. However, asymptotic formulas can be derived as shown by Elizalde and Noy in [7]. They provide an estimation of $\alpha_n(\sigma)$ for any pattern σ of length 3 and also for some patterns of length 4. Nowadays, an asymptotic formula of $\alpha_n(\sigma)$ is not even known for all the patterns of length 4.

Elizalde in [5] showed that for any $\sigma \in \mathcal{S}_m$, the following limit exists,

$$\rho_\sigma = \lim_{n \rightarrow \infty} \left(\frac{\alpha_n(\sigma)}{n!} \right)^{1/n},$$

and that $0.7839 < \rho_\sigma < 1$. In particular, it is known that $\alpha_n(\sigma) \sim c \rho_\sigma^n n!$, for some constant c that depends on σ .

Whereas it is very hard to exactly compute ρ_σ for any $\sigma \in \mathcal{S}_m$, it is possible to provide upper and lower bounds in terms of m . Besides, it is interesting to see which patterns are extremal in that sense. A pattern of length m is called *monotone* if it is either $(12\dots m)$ or $(m\dots 21)$. It is clear that $\alpha_n(12\dots m) = \alpha_n(m\dots 21)$, since $\pi \in \mathcal{S}_n$ is $(12\dots m)$ -avoiding if and only if its reversing $\tilde{\pi} = (\pi_n, \dots, \pi_1)$ is $(m\dots 21)$ -avoiding. In [7] it was conjectured that monotone patterns are the most avoided ones among all patterns of length m , when n is large enough. This is known as the *Consecutive Monotone Pattern (CMP) Conjecture*.

Conjecture 1 (CMP conjecture [7]). For any $\sigma \in \mathcal{S}_m$,

$$\rho_\sigma \leq \rho_{(12\dots m)}.$$

The results in [7], determining ρ_σ for any $\sigma \in \mathcal{S}_3$ settle in the affirmative the CMP conjecture for patterns of length 3. Elizalde and Noy in [8] show that the conjecture is true for the class of non-overlapping patterns. They also study the monotone pattern and provide the exact value of $\rho_{(12\dots m)}$ implicitly as the smallest root of a formal power series.

Regarding the least avoided pattern among all the patterns of length m , Nakamura in [13] posed the following conjecture,

Conjecture 2 ([13]). For any $\sigma \in \mathcal{S}_m$,

$$\rho_\sigma \geq \rho_{(12\dots m-2, m, m-1)}.$$

Both conjectures have been recently proved by Elizalde in [6]. The proofs are based on computing the

generating function for the number of σ -avoiding permutations, $P_\sigma(z) = \sum \alpha_n(\sigma) \frac{z^n}{n!}$, combined with the cluster method of Goulden and Jackson [9].

Here we will use a complete different approach to the consecutive pattern avoiding problem through the so called probabilistic method. While this approach is not as precise as the generating function technique, it is simpler. This means that it provides more direct proofs of some existing results, such as the CMP conjecture, and indeed, it allows us to go further in some directions, as will be seen in Section 5.

Our first result bounds from above ρ_σ when the pattern σ is not monotone.

Theorem 3. *For any $\sigma \in \mathcal{S}_m \setminus \{(12 \dots m), (m \dots 21)\}$,*

$$\rho_\sigma \leq 1 - \frac{1}{m!} + O\left(\frac{1}{m^2 m!}\right).$$

The proof of this and all the following results, have a probabilistic flavor. We set out the problem through the Poisson Paradigm (see e.g. [2]) which asserts that, in a probability space, events that are nearly independent should behave similar as if they were so. To prove this theorem we make use of the Suen's Inequality [15], a powerful tool that provides an upper bound on the probability that some events do not happen at the same time.

Theorem 3 can be extended to the whole set of patterns, \mathcal{S}_m , by weakening the upper bound: for any $\sigma \in \mathcal{S}_m$,

$$\rho_\sigma \leq 1 - \frac{1}{m!} + O\left(\frac{1}{m \cdot m!}\right),$$

however, this bound is not strong enough to prove the CMP conjecture. From the results given in [8], one can derive a lower bound on $\rho_{(12 \dots m)}$ to show that the CMP conjecture holds for any large enough m . A more careful analysis on the constants hidden inside the asymptotic notation shows that it is enough to consider $m \geq 5$.

The second part of the article is devoted to give a general lower bound on ρ_σ when $\sigma \in \mathcal{S}_m$.

Theorem 4. *For any $\sigma \in \mathcal{S}_m$,*

$$\rho_\sigma \geq 1 - \frac{1}{m!} - O\left(\frac{m-1}{(m!)^2}\right).$$

To prove this lower bound we use a one-sided version of the Lovász Local Lemma (see [14]). This bound is asymptotically tight and an extremal example is provided by the pattern $(12 \dots m-2, m, m-1)$. Unlike in the case of the upper bound and the CMP conjecture, the proof of Theorem 4 can not be adapted to extract a proof of Conjecture 2.

As Theorem 3 and Theorem 4 give bounds for the value of ρ_σ in terms of m , a natural question is to determine how do most of the patterns behave. In this direction a much stronger upper bound, close to the general lower bound, is shown to hold for most of the patterns.

Theorem 5. *Let $\sigma \in \mathcal{S}_m$ chosen uniformly at random. Then, for any $2 \leq k \leq m/2$,*

$$\rho_\sigma \leq 1 - \frac{1}{m!} + O\left(\frac{4^m}{(m-k)!m!}\right),$$

with probability at least $1 - \frac{2}{(k+1)!} - m2^{-m/2}$.

This theorem shows that most of the patterns behave similar to the least avoided one. The idea behind this result is that the number of permutations avoiding a pattern depends on the maximum overlapping position of this pattern. It can be shown that almost all patterns do not have a large overlap and thus, they are far from the upper bound attained by monotone patterns, the ones with maximum overlap.

This paper is organized as follows. In Section 2, Theorem 3 is proven. A lower bound on $\rho_{(12\dots m)}$ is derived in Section 3 completing the proof of the CMP conjecture. Section 4 is devoted to the proof of Theorem 4. Finally, in Section 5 we provide the proof of Theorem 5.

2 An upper bound on ρ_σ

Consider the set of events $\mathcal{A} = \{A_1, \dots, A_N\}$ with associated indicator random variables X_1, \dots, X_N and let $X = \sum_{i=1}^N X_i$. In general, the events in \mathcal{A} will be considered to be bad and the aim is to bound, either from above or from below, the probability that none of these bad events occurs. We will denote by μ the expected number of bad events, this is, $\mu = \mathbb{E}(X) = \sum_{i=1}^N \Pr(A_i)$.

A *dependency graph* of \mathcal{A} is a graph H with vertex set $V(H) = \{1, \dots, N\}$ where if two disjoint subsets $S, T \subseteq [N]$ share no edges then $\{A_i\}_{i \in S}$ and $\{A_j\}_{j \in T}$ are independent.

Two parameters are defined to control the dependencies among all events. To measure the global effect of the dependencies, consider

$$\Delta = \sum_{ij \in E(H)} \Pr(A_i \cap A_j),$$

and for the local one,

$$\delta = \max_{1 \leq i \leq N} \sum_{j: ij \in E(H)} \Pr(A_j),$$

where $E(H)$ denotes the edge set of H .

We will use the following version of Suen's inequality (see e.g. Theorem 2 in [10]),

Theorem 6 (Suen's inequality). *With the above notation,*

$$\Pr(X = 0) = \Pr(\cap \overline{A_i}) \leq \exp \left\{ - \left(1 - \frac{\Delta e^{2\delta}}{\mu} \right) \mu \right\}. \quad (1)$$

Suen's inequality bounds from above the probability of having no bad events in terms of the expected number of events, but also takes into account the pairwise dependence of the events. Thus, if the dependencies among the events are weak or unlikely, we will be able to give a meaningful upper bound on such probability.

Let $\pi \in \mathcal{S}_n$ chosen uniformly at random, and let $\sigma \in \mathcal{S}_m$ be a fixed pattern. For any $0 \leq i \leq n - m$ we define the event $A_i := \{\text{st}(\pi_{i+1}, \dots, \pi_{i+m}) = \sigma\}$. Then π avoids σ as a consecutive pattern if and only if $X = 0$, this is, no copy of the pattern σ appears. By computing the probability of this event,

$$\alpha_n(\sigma) = \Pr(X = 0)n! ,$$

where $\Pr(X = 0)$ depends on σ . In particular we will be interested in

$$\rho_\sigma = \lim_{n \rightarrow \infty} \Pr(X = 0)^{1/n} . \quad (2)$$

Bounding from above the number of edges in a dependency graph H is crucial in order to give a proper upper bound on the probability that any of the events happens at the same time. The following lemma shows that there are many pairs of sets of events that share no edges.

Lemma 7. *Let $S, T \subseteq \{0, 1, \dots, n - m\}$ be two disjoint subsets of indexes such that for any $i \in S$ and any $j \in T$, we have $|i - j| \geq m$. Then, the events $\{A_i\}_{i \in S}$ and $\{A_j\}_{j \in T}$ are independent.*

Proof. For any two disjoint sets $U^+, U^- \subseteq \{0, 1, \dots, n - m\}$, define the event

$$A_{U^+, U^-} := \left\{ \bigcap_{i \in U^+} A_i \wedge \bigcap_{i \in U^-} \overline{A_i} \right\} .$$

It suffices to show that for any two disjoint subsets $S^+, S^- \subseteq S$ and $T^+, T^- \subseteq T$,

$$\Pr(A_{S^+, S^-} \mid A_{T^+, T^-}) = \Pr(A_{S^+, S^-}) . \quad (3)$$

We say that $j \in \{0, 1, \dots, n - m\}$ belongs to the support of S , $\text{supp}(S)$, if there exists $i \in S$ such that $0 \leq j - i \leq m - 1$. Observe that $\text{supp}(S) \cap \text{supp}(T) = \emptyset$, by the assumptions on S and T . Clearly, the event A_{S^+, S^-} is determined by the elements appearing in the positions indexed by $\text{supp}(S)$.

Denote by $\mathcal{T} \subseteq \mathcal{S}_n$ the subset of permutations of length n that satisfies A_{T^+, T^-} . Choose $\tau \in \mathcal{T}$ uniformly at random. It is enough to consider τ restricted on $\text{supp}(S)$, τ' , and show that its standardization, $\text{st}(\tau')$, is uniformly distributed in $\mathcal{S}_{|\text{supp}(S)|}$.

The key observation is that A_{T^+, T^-} might condition which elements lie in $\text{supp}(S)$ but does not impose anything on their order. The event A_{T^+, T^-} makes no direct restriction affecting the order of the elements in $\text{supp}(S)$. Therefore, the elements appearing in τ' may be conditioned by A_{T^+, T^-} , but $\text{st}(\tau')$ is not affected by A_{T^+, T^-} . Since A_{S^+, S^-} is satisfied in τ' if and only if, it is satisfied in $\text{st}(\tau')$ (with the corresponding relabeling), equation (3) holds. \square

The previous lemma suggests that a good dependency graph for the set of events \mathcal{A} is the circulant graph H with vertex set $V(H) = \{0, 1, \dots, n - m\}$, where $ij \in E(H)$ if and only if $0 < |i - j| < m$. Throughout the paper, we will use the former circulant graph as a dependency graph of \mathcal{A} .

A simple upper bound follows directly from the previous observation. Consider $I = \{km : 0 \leq k < n/m\}$, then

$$\Pr(X = 0) = \Pr\left(\bigcap_{i=0}^{n-m} \overline{A_i}\right) \leq \Pr\left(\bigcap_{i \in I} \overline{A_i}\right) = \prod_{i \in I} \left(1 - \Pr\left(A_i \mid \bigcap_{j \in I, j < i} \overline{A_j}\right)\right).$$

By using Lemma 7 with $S = \{i\}$ and $T = \{j : j \in I, j < i\}$,

$$1 - \Pr\left(A_i \mid \bigcap_{j \in I, j < i} \overline{A_j}\right) = 1 - \Pr(A_i) = 1 - \frac{1}{m!}.$$

Since $|I| \leq n/m$, this implies

$$\rho_\sigma \leq \left(1 - \frac{1}{m!}\right)^{1/m} = 1 - O\left(\frac{1}{m \cdot m!}\right).$$

However, a better bound is given in Theorem 3 by taking into account the interaction between pairs of dependent events.

A pattern $\sigma \in \mathcal{S}_m$ has an *overlap at k* if $\text{st}(\sigma_1, \dots, \sigma_k) = \text{st}(\sigma_{m-k+1}, \dots, \sigma_m)$, this is, the first and the last k positions have the same relative order. If a pattern does not have an overlap at k , then

$$\Pr(A_i \cap A_{i+m-k}) = 0. \tag{4}$$

For any $1 \leq k \leq m-1$, define the set $\mathcal{M}_k \subseteq \mathcal{S}_m$ as the set of patterns of length m that have no overlap larger than k . The elements in \mathcal{M}_1 are called *non-overlapping* patterns. They have been enumerated in [4] and also extensively studied in [6].

Observe that \mathcal{M}_{m-1} is the whole set of patterns of length m . One of the crucial facts to prove Theorem 3 is to show that $\mathcal{M}_{m-1} \setminus \mathcal{M}_{m-2}$, the set of patterns that have an overlap at $m-1$, only consists of the monotone patterns.

Lemma 8. *For any $m \geq 3$,*

$$\mathcal{M}_{m-1} \setminus \mathcal{M}_{m-2} = \{(12 \dots m), (m \dots 21)\}.$$

Proof. It is clear that both monotone patterns belong to $\mathcal{M}_{m-1} \setminus \mathcal{M}_{m-2}$. Let us show that any other $\sigma \in \mathcal{S}_m \setminus \{(12 \dots m), (m \dots 21)\}$ does not. Suppose that $\sigma \in \mathcal{M}_{m-1} \setminus \mathcal{M}_{m-2}$. This implies that

$$\text{st}(\sigma_1 \dots \sigma_{m-1}) = \text{st}(\sigma_2 \dots \sigma_m). \tag{5}$$

Since σ is not a monotone pattern, there exists an index $2 \leq i \leq m-1$ such that $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$ or $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$. Without loss of generality we assume the latter. Now observe that (5) implies that if $\sigma_{i-1} < \sigma_i$, then $\sigma_i < \sigma_{i+1}$, leading a contradiction. \square

Thus, we can consider that the maximum overlap of a pattern $\sigma \in \mathcal{S}_m \setminus \{(12 \dots m), (m \dots 21)\}$ is at most at $m-2$. This can not be improved since there are non monotone patterns that have an overlap at $m-2$. For instance, consider $m = 2t$ and $\sigma = (1, t+1, 2, t+2, \dots, t, 2t)$.

The following lemma gives some insight of the structure of the permutations that contain two given occurrences of a pattern σ .

Lemma 9. *Let $\sigma \in \mathcal{S}_m$ be a pattern with an overlap at k and suppose that $\tau \in \mathcal{S}_{2m-k}$ is such that the events A_0 and A_{m-k} hold. If $\sigma' = \text{st}(\sigma_{m-k+1}, \dots, \sigma_m)$, then, for any $0 \leq i < k$, we have $\tau_{m-i} = \sigma_{k-i} + \sigma_{m-i} - \sigma'_{k-i}$.*

Proof. Fix some $i < k$. By the event A_0 , we know that τ_{m-i} should be larger than $\sigma_{m-i} - 1$ elements and smaller than $m - \sigma_{m-i}$ elements from $(\tau_1, \dots, \tau_{m-i-1}, \tau_{m-i+1}, \dots, \tau_m)$. By the event A_{m-k} , it is also true that τ_{m-i} is larger than $\sigma_{k-i} - 1$ and smaller than $m - \sigma_{k-i}$ elements from $(\tau_{m-k+1}, \dots, \tau_{m-i-1}, \tau_{m-i+1}, \dots, \tau_{2m-k})$.

Consider now the permutation $\sigma' = \text{st}(\sigma_{m-k+1}, \dots, \sigma_m) \in \mathcal{S}_k$. Then there are $\sigma'_{k-i} - 1$ elements that are counted twice when we look at the elements smaller than σ_{m-i} or σ_{k-i} , and $k - \sigma'_{k-i}$ also double counted when we look to the larger ones. Therefore

$$\tau_{m-i} > \sigma_{k-i} + \sigma_{m-i} - 2 - (\sigma'_{k-i} - 1),$$

and

$$\tau_{m-i} \leq 2m - k - (m - \sigma_{k-i} + m - \sigma_{m-i} - (k - \sigma'_{k-i})).$$

Observing that the first inequality is strict,

$$\tau_{m-i} = \sigma_{i+1} + \sigma_{m-i} - \sigma'_{k-i}.$$

\square

Using this last lemma, we can provide an upper bound on the probability that two given occurrences of a pattern appear.

Lemma 10. *For any $\sigma \in \mathcal{S}_m$ and any $1 \leq k \leq m-1$,*

$$\Pr(A_i \wedge A_{i+m-k}) \leq \frac{4^{m-k}}{\sqrt{\pi(m-k)} \cdot (2m-k)!}.$$

Proof. If σ does not have an overlap at k , $\Pr(A_i \wedge A_{i+m-k}) = 0$ and we are done. Thus, assume that σ has an overlap at k .

Set $\tau = \text{st}(\pi_{i+1}, \dots, \pi_{i+2m-k})$. Recall that $\pi \in \mathcal{S}_n$ has been chosen uniformly at random, which implies that τ is uniformly distributed in \mathcal{S}_{2m-k} . Moreover, π satisfies A_i and A_{i+m-k} if and only if τ satisfies A_0 and A_{m-k} .

There are $(2m-k)!$ possible candidates for τ . We will count how many of them are such that the events A_0 and A_{m-k} hold. By Lemma 9, we know that the elements $(\tau_{m-k+1}, \dots, \tau_m)$ are uniquely determined by σ and k . Thus, one must select a subset of $m-k$ elements among the $2m-2k$ available ones, to construct $(\tau_1, \dots, \tau_{m-k})$. Since τ satisfies A_0 , once these elements have been chosen, there is just one order such that $\text{st}(\tau_1, \dots, \tau_m) = \sigma$, and only one way to set the last $m-k$ elements of τ , in order to satisfy A_{m-k} .

Hence, for $\pi \in \mathcal{S}_n$,

$$\Pr(A_i \wedge A_{i+m-k}) \leq \frac{\binom{2(m-k)}{m-k}}{(2m-k)!} \leq \frac{4^{m-k}}{\sqrt{\pi(m-k)} \cdot (2m-k)!}.$$

where we have used that $\binom{2a}{a} \leq \frac{4^a}{\sqrt{\pi a}}$. One can prove this last inequality by using Stirling's approximation. \square

Now we are able to prove the main theorem.

Proof of Theorem 3. First of all we compute μ , Δ and δ , needed to apply Suen's inequality. The expected number of occurrences of the pattern σ does not depend on σ and can be computed as

$$\mu = \sum_{i=0}^{n-m} \Pr(A_i) = \frac{n-m+1}{m!} \leq \frac{n}{m!}.$$

Recall that by the choice of the dependency graph H (inspired by Lemma 7) two events A_i and A_j share no edge if $|i-j| \geq m$. Assume that $i < j$ and $j-i = m-k$, then by Lemma 10,

$$\Pr(A_i \wedge A_j) \leq \frac{4^{m-k}}{\sqrt{2\pi}(2m-k)!} \quad \text{if } k \leq m-2.$$

and, since σ is not monotone, by (4) and Lemma 8,

$$\Pr(A_i \wedge A_j) = 0 \quad \text{if } k = m-1.$$

Hence,

$$\begin{aligned} \sum_{j=i+1}^{i+m-1} \Pr(A_i \wedge A_j) &\leq \sum_{k=1}^{m-2} \frac{4^{m-k}}{\sqrt{2\pi}(2m-k)!} \\ &= \left(1 + \frac{4}{m+3} + O(m^{-2})\right) \frac{16}{\sqrt{2\pi}(m+2)!} \leq \frac{17}{\sqrt{2\pi}(m+2)!}, \end{aligned} \tag{6}$$

for any m large enough.

Then, Δ can be expressed as

$$\Delta = \sum_{0=i}^{n-m} \sum_{j=i+1}^{i+m-1} \Pr(A_i \wedge A_j) \leq \frac{17n}{\sqrt{2\pi}(m+2)!}.$$

Since the degree of a vertex in the dependency graph H is at most $2(m-1)$,

$$\delta = \max_{0 \leq i \leq n-m} \sum_{j: ij \in E(H)} \Pr(A_j) = 2(m-1) \Pr(A_j) = \frac{2(m-1)}{m!} \leq \frac{2}{(m-1)!}.$$

Using that $e^{2\delta} \leq e^{4/(m-1)!} \leq 2$ if $m \geq 4$, Suen's inequality (see (2)) implies that for a large enough m

$$\begin{aligned} \rho_\sigma &\leq \exp \left(- \frac{1 - \frac{34}{\sqrt{2\pi}(m+2)(m+1)}}{m!} \right) \\ &\leq 1 - \frac{\frac{1}{m!} - \frac{34}{\sqrt{2\pi}(m+2)(m+1)m!}}{1 + \frac{1}{m!}} \\ &\leq 1 - \left(1 - O\left(\frac{1}{m!}\right) \right) \left(\frac{1}{m!} - \frac{34}{\sqrt{2\pi}(m+2)(m+1)m!} \right) \\ &\leq 1 - \frac{1}{m!} + \frac{14}{m^2 m!}. \end{aligned}$$

for any large enough m . We have used that $e^{-a} \leq 1 - \frac{a}{1+a}$, for any $a \geq 0$. \square

3 A probabilistic proof of CMP conjecture

In this section we aim to provide an alternative proof of the CMP conjecture. We do it by obtaining a lower bound on $\rho_{(12\dots m)}$ and showing that this bound is larger than the upper bound obtained in Theorem 3. A recent result of Elisalde and Noy gives an implicit expression for $\rho_{(12\dots m)}$.

Theorem 11 (Elisalde and Noy [8]). *Let $z_0 = \rho_{(12\dots m)}^{-1}$, then z_0 is the smallest solution of*

$$g(z) = \sum_{i \geq 0} \frac{z^{mi}}{(mi)!} - \sum_{i \geq 0} \frac{z^{mi+1}}{(mi+1)!}.$$

From this last theorem we can extract an explicit lower bound on $\rho_{(12\dots m)}$.

Lemma 12. *For any m large enough,*

$$\rho_{(12\dots m)} \geq 1 - \frac{1}{m!} + \frac{1}{m \cdot m!} + O\left(\frac{1}{m^2 \cdot m!}\right).$$

Proof. Observe that

$$f(z) = 1 - z + \frac{z^m}{m!} - \frac{z^{m+1}}{(m+1)!} + \frac{z^{2m}}{(2m)!} \geq g(z),$$

since $g(z)$ is an alternating sum whose terms are strictly decreasing. Since $g(0) = 1$ and z_0 is the smallest root of $g(z)$ we can conclude that z_1 , the smallest root of $f(z)$, is at least z_0 . Thus $\rho_{(12\dots m)} \geq 1/z_1$ and it suffices to compute an upper bound on z_1 .

Write $z = (1 - \varepsilon)^{-1}$, then $z^{-2m}f(z) = 0$ becomes

$$-(1 - \varepsilon)^{2m-1}\varepsilon + \frac{(1 - \varepsilon)^{m-1}}{(m+1)!}(m - (m+1)\varepsilon) + \frac{1}{(2m)!} = 0.$$

Using $1 - nx \leq (1 - x)^n \leq 1 - nx + n^2x^2$,

$$\begin{aligned} 0 &\leq -(1 - (2m-1)\varepsilon)\varepsilon + \frac{1 - (m-1)\varepsilon + (m-1)^2\varepsilon^2}{(m+1)!}(m - (m+1)\varepsilon) + \frac{1}{(2m)!} \\ &\leq \left(2m-1 + \frac{(m-1)(m^2+1)}{(m+1)!}\right)\varepsilon^2 - \left(1 + \frac{m^2+1}{(m+1)!}\right)\varepsilon + \left(\frac{m}{(m+1)!} + \frac{1}{(2m)!}\right). \end{aligned} \quad (7)$$

Let ε' be the solution of the last equation with equality. Then, $\rho_{(12\dots m)} \geq (1 - \varepsilon')$. If m is large enough we can get an asymptotic expression for ε' . Suppose that $b^2 \gg 4ac$, then the smallest solution of $ax^2 + bx + c = 0$ can be approximated by

$$x = -\frac{c}{b} - \frac{ac^2}{b^3} + O\left(\frac{a^2c^3}{b^5}\right). \quad (8)$$

This leads to

$$\begin{aligned} \varepsilon' &= \frac{\frac{m}{(m+1)!} + \frac{1}{(2m)!}}{1 + \frac{m^2+1}{(m+1)!}} + O\left(\frac{m^3}{(m+1)!^2}\right) \\ &= \frac{m}{(m+1)!} + O\left(\frac{m^3}{(m+1)!^2}\right) \\ &= \frac{1}{m! \left(1 + \frac{1}{m}\right)} + O\left(\frac{m^3}{(m+1)!^2}\right) \\ &= \frac{1}{m!} - \frac{1}{m \cdot m!} + O\left(\frac{1}{m^2 \cdot m!}\right), \end{aligned}$$

where we have used $(1+x)^{-1} = 1 - x + O(x^2)$ in the second and the last inequalities. This proves the lemma. \square

The CMP conjecture comes as an straightforward corollary of Theorem 3 together with Lemma 12.

Corollary 13. *For any large enough m , the CMP conjecture is true. Moreover, for any $\sigma \in \mathcal{S}_m \setminus \{(12 \dots m), (m \dots 21)\}$,*

$$1 - \rho_\sigma \geq \left(1 + \frac{1}{m}\right) (1 - \rho_{(12 \dots m)}) .$$

Thus, this corollary does not only show that the CMP conjecture is true, but also provides a lower estimation of the minimum gap between $\rho_{(12 \dots m)}$ and ρ_σ , for any $\sigma \in \mathcal{S}_m \setminus \{(12 \dots m), (m \dots 21)\}$.

Note that the last corollary holds for any m large enough. An upper bound without any assumption on m that can be derived from (1) and (6) in Theorem 3. Comparing this bound with the exact lower bound that follows from (7) in Lemma 8, one can check that the CMP conjecture holds for any $m \geq 5$.

As we will see in the next section, it is also possible to provide a lower bound on ρ_σ in terms of m without using generating functions. Unlike the upper bound case, the lower bound just takes into account the number of dependencies among the events, but not the nature of these dependencies. It might be interesting to give a direct proof of the lower bound for the monotone pattern (Lemma 8), which does not rely upon any other result. For such a purpose it would be useful to understand the probabilities $\Pr \left(A_i \mid \bigcap_{j < i} \overline{A_j} \right)$ when $\sigma = (12 \dots m)$.

4 A lower bound on ρ_σ .

The setting used to give an upper bound to the number of permutations avoiding a given pattern can be also used to provide a lower bound on ρ_σ . Now we need a way to bound from below the probability that $X = 0$ and for such a purpose we will use the Lovász Local Lemma.

Usually, the Local Lemma is used to show the existence of a certain configuration that does not satisfy any of the bad events in \mathcal{A} . However, in our problem it is trivial to see that for any pattern $\sigma \in \mathcal{S}_m$ there exists at least one permutation of length n that avoids σ . Nevertheless, it also provides an explicit lower bound on the probability that such configuration exists, giving a lower estimation on the number of such configurations. We will use it to derive a lower bound on the number of permutations of length n that avoid σ .

The following version of the Local Lemma was proposed by Peres and Schlag in [14] and it is convenient for our approach.

Lemma 14 (One-sided Local Lemma). *Let x_1, x_2, \dots, x_N be a sequence of numbers in $(0, 1)$. Assume that, for every $i \in N$, there is an integer $0 < m(i) \leq i$ such that*

$$\Pr \left(A_i \mid \bigcap_{j < m(i)} \overline{A_j} \right) \leq x_i \prod_{k=m(i)}^{i-1} (1 - x_k) . \quad (9)$$

Then,

$$\Pr(X = 0) = \Pr \left(\bigcap_{i=1}^N \overline{A_i} \right) \geq \prod_{i=1}^N (1 - x_i) . \quad (10)$$

To use the Local Lemma a dependency graph on the set of events must be set. In the case of the one-sided version, the graph is defined implicitly in (9) as the directed circulant graph with out-degree $i - m(i)$. Thus, the same dependency graph used for Suen's inequality is also valid to apply the Local Lemma.

Next, we give the proof of the lower bound on ρ_σ .

Proof of Theorem 4. Let $\mathcal{A} = \{A_0, \dots, A_{n-m}\}$ and X be defined as in Section 2. Set $m(i) = i - m + 1$. Using Lemma 7 with $S = \{i\}$ and $T = \{0, 1, \dots, i - m\}$

$$\Pr \left(A_i \mid \bigcap_{j \leq i-m} \overline{A_j} \right) = \Pr(A_i) . \quad (11)$$

Since all the events are symmetric we set $x_i = x$, for any $0 \leq i \leq n - m$. Then, condition (9) becomes

$$\Pr(A_i) \leq x(1 - x)^{m-1} . \quad (12)$$

Recall that $\Pr(A_i) = \frac{1}{m!}$. Thus, the previous equation implies that $x > \frac{1}{m!}$. Besides, we are interested on keeping x as small as possible, because of (10). Let us write $x = \frac{e^{f(m)}}{m!}$ for some positive function $f(m)$. Hence, using $(1 - x) \leq e^{-x}$, condition (12) implies

$$\frac{e^{f(m)}(m-1)}{m!} \leq f(m) ,$$

which also implies $f(m) \geq \frac{m-1}{m!}$, since $f(m) \geq 0$. By setting $x = \frac{e^{\frac{m-1}{m!}}}{m!}$, condition (9) is satisfied and the Local Lemma can be applied. In particular, we obtain the following lower bound on the probability that $X = 0$,

$$\Pr(X = 0) = \Pr \left(\bigcap_{i=0}^{n-m} \overline{A_i} \right) \geq \left(1 - \frac{e^{(m-1)/m!}}{m!} \right)^{n-m+1} .$$

and using (2),

$$\rho_\sigma \geq 1 - \frac{e^{(m-1)/m!}}{m!} = 1 - \frac{1}{m!} - O \left(\frac{m-1}{(m!^2)} \right) .$$

□

The lower bound given by Theorem 4 is tight. This can be shown using a result of Elizalde in [6], where the author proved that the least avoided pattern is $(12 \dots m-2, m, m-1)$. The author also gives an implicit lower bound to $z_0 = \rho_{(12 \dots m-2, m, m-1)}^{-1}$ as the smallest root of

$$f(z) = 1 - z + \frac{z^m}{m!} - m \frac{z^{2m+1}}{(2m-1)!} .$$

An explicit upper bound can be derived from the previous equation, as in Lemma 8.

$$\rho_{(12\dots m-2, m, m-1)} \leq 1 - \frac{1}{m!} - O\left(\frac{m-1}{(m!)^2}\right).$$

In order to prove Conjecture 2, one could try to use the same strategy we have used for the CMP conjecture. First, determine the subset of patterns σ such that $\alpha_n(\sigma) = \alpha_n(12\dots m-2, m, m-1)$ and finally, improve the lower bound for the patterns which are not in the previous subset. However, this approach is hopeless to tackle Conjecture 2. Notice that no assumption on the properties of the pattern has been used in the proof of the lower bound, like in the proof of the upper bound in Theorem 3. Unfortunately, the Local Lemma can not distinguish the different nature of the dependencies among events. Thus, no better lower bound can be achieved by restricting to a smaller subset of patterns. This is also the main problem to prove Lemma 8 using our approach.

In the next section we will improve the upper bound of Theorem 3 for large subsets of patterns.

5 The typical behavior of patterns.

The results of the previous sections provide tight upper and lower bounds on ρ_σ for any $\sigma \in \mathcal{S}_m$. In this section we want to show that, for a typical pattern, ρ_σ lies much closer to the lower bound than to the upper bound. This is, the number of σ -avoiding permutations of length n , when $\sigma \in \mathcal{S}_m$ chosen uniformly at random, is closer to the number of permutations that avoid $(12\dots m-2, m, m-1)$ than to the number of permutations that avoid $(12\dots m)$.

Define $\mathcal{N}_k \subseteq \mathcal{S}_m$ as the set of patterns of length m that overlap at position k . The following lemma bounds from above the size of these sets.

Lemma 15. *Let $\sigma \in \mathcal{S}_m$ chosen uniformly at random, then*

1. $\Pr(\sigma \in \mathcal{N}_k) = \frac{1}{k!}$ if $2 \leq 2k \leq m$.
2. $\Pr(\sigma \in \mathcal{N}_k) \leq 2^{-m/2}$ if $m < 2k \leq 2(m-1)$.

Proof of 1. Choose $\sigma \in \mathcal{S}_m$ uniformly at random. Recall that the condition for $\sigma \in \mathcal{N}_k$ is that $\tau^1 = \text{st}(\sigma_1, \dots, \sigma_k)$ and $\tau^2 = \text{st}(\sigma_{m-k+1}, \dots, \sigma_m)$ are equal. If $2k \leq m$, then τ^1 and τ^2 are independent by Lemma 7 and uniformly distributed in \mathcal{S}_k . For any $\tau, \tau' \in \mathcal{S}_k$

$$\Pr(\tau^1 = \tau \mid \tau^2 = \tau') = \Pr(\tau^1 = \tau).$$

Thus, we can compute the exact probability of being in \mathcal{N}_k

$$\Pr(\sigma \in \mathcal{N}_k) = \Pr(\tau^1 = \tau^2) = \sum_{\tau \in \mathcal{S}_k} \Pr(\tau^1 = \tau \wedge \tau^2 = \tau) = k! \Pr(\tau^1 = \tau)^2 = \frac{1}{k!}.$$

□

Proof of 2. Choose $\sigma \in \mathcal{S}_m$ uniformly at random. Partition the pattern σ in parts of length $m - k$ by defining $\tau^i = \text{st}(\sigma_{(m-k)(i-1)+1}, \dots, \sigma_{(m-k)i})$ for any $1 \leq i \leq \lfloor \frac{m}{m-k} \rfloor$. Observe that, in order to have an overlap at k we must have $\tau_1 = \tau_i$ for any $i > 1$. This condition is clearly necessary but not sufficient for a pattern to overlap at k .

Since $2k > m$, we have at least $\lfloor \frac{m}{m-k} \rfloor \geq 2$ parts. By the choice of σ , the permutations τ^i are uniformly distributed in \mathcal{S}_{m-k} , and by Lemma 7, they are mutually independent. This implies,

$$\Pr(\sigma \in \mathcal{N}_k) \leq \prod_{i>1} \Pr(\tau_i = \tau_1) = \left(\frac{1}{(m-k)!} \right)^{\lfloor \frac{m}{m-k} \rfloor - 1} \leq 2^{-m/2+1},$$

for any $k \leq m - 2$. If $k = m - 1$, \mathcal{N}_{m-1} is the set of patterns with an overlap at $m - 1$ and the upper bound is directly implied by Lemma 8. \square

Unlike in the case when $k \leq 2m$, where we can determine exactly the size of \mathcal{N}_k , a non tight upper bound is given when $k > 2m$. Observe that the sets \mathcal{N}_k cover all \mathcal{S}_m but they are not a partition of it. For instance, monotone patterns belong to all such sets, since they overlap at any possible position. However, we conjecture that

$$|\mathcal{N}_k| \leq \frac{1}{k!},$$

for every $1 \leq k \leq m - 1$.

We use the previous lemma to give a lower bound on the size of \mathcal{M}_k , the set of patterns that have no overlap at any positions larger than k .

Lemma 16. *Let $\sigma \in \mathcal{S}_m$ chosen uniformly at random. Then, for any $1 \leq k \leq m/2$,*

$$\Pr(\sigma \in \mathcal{M}_k) \geq 1 - \frac{2}{(k+1)!} - m2^{-m/2}.$$

Proof. Observe that we can bound from below the size of \mathcal{M}_k using the sets \mathcal{N}_k ,

$$|\mathcal{M}_k| = \left| \mathcal{S}_m \setminus \bigcup_{\ell=k+1}^{m-1} \mathcal{N}_\ell \right| \geq m! - \sum_{\ell=k+1}^{m-1} |\mathcal{N}_\ell|. \quad (13)$$

By Lemma 15, for any k such that $2k \leq m$,

$$\sum_{\ell=k+1}^{m-1} \Pr(\sigma \in \mathcal{N}_\ell) \leq \frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \dots + \frac{1}{\lfloor m/2 \rfloor!} + \frac{m}{2} 2^{-m/2+1}.$$

Using the relation in (13) gives

$$\Pr(\sigma \in \mathcal{M}_k) \geq 1 - \sum_{\ell=k+1}^{m-1} \Pr(\sigma \in \mathcal{N}_\ell) \geq 1 - \sum_{\ell=k+1}^{m/2} \frac{1}{\ell!} - m2^{-m/2} \geq 1 - \frac{2}{(k+1)!} - m2^{-m/2}.$$

\square

Recall that \mathcal{M}_1 corresponds to the set of non-overlapping patterns. The proof of Lemma 16 implies that $|\mathcal{M}_1| \geq (3 - e)m!$. This bound can be refined. Indeed, Bóna [4] showed that

$$0.364098149 \leq \frac{|\mathcal{M}_1|}{m!} \leq 0.3640992743 .$$

The previous bound on $|\mathcal{M}_k|$ is clearly non sharp. A better estimation of the size of \mathcal{N}_k when $2k > m$, would help to understand the distribution of ρ_σ when $\sigma \in \mathcal{S}_m$ is chosen uniformly at random.

Next lemma shows that a better bound on Δ can be given if the pattern does not have a large overlap.

Lemma 17. *For any $\sigma \in \mathcal{M}_k$,*

$$\Delta \leq \frac{4^{m-k}}{(2m-k)!} n .$$

Proof. Since $\sigma \in \mathcal{M}_k$ we have $\Pr(A_i \wedge A_{i+m-j}) = 0$ for any j such that $k < j \leq m-1$. Using Lemma 10

$$\begin{aligned} \Delta &\leq \sum_{i=0}^{n-m} \sum_{j=1}^k \Pr(A_i \wedge A_{i+m-j}) \\ &\leq n \sum_{j=1}^k \frac{4^{m-j}}{\sqrt{\pi(m-j)}(2m-j)!} \\ &\leq \frac{4^{m-k}}{(2m-k)!} n . \end{aligned}$$

□

Proof of Theorem 5. Assume that $\sigma \in \mathcal{M}_k$. It follows from Lemma 17 that

$$\frac{\Delta}{\mu} \leq \frac{4^{m-k} m!}{(2m-k)!} = \frac{4^m}{\binom{2m-k}{m} (m-k)!} \leq \frac{4^m}{(m-k)!} .$$

Since $e^{2\delta} \leq e^{4/(m-1)!} \leq 2$ for any $m \geq 4$, using (1) we can derive the following upper bound,

$$\Pr(X = 0) \leq \exp \left(- \frac{1 - O\left(\frac{4^m}{(m-k)!}\right)}{m!} n \right) .$$

From (2),

$$\rho_\sigma = \lim_{n \rightarrow \infty} \Pr(X = 0)^{1/n} \leq 1 - \frac{1}{m!} + O\left(\frac{4^m}{(m-k)!m!}\right) ,$$

where we have used $e^{-a} \leq 1 - \frac{a}{1+a}$.

This upper bound holds when $\sigma \in \mathcal{M}_k$, and this holds with probability at least $1 - \frac{2}{(k+1)!} - m2^{-m/2}$ when σ is chosen uniformly at random, by Lemma 16. □

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